$$\Rightarrow \frac{2}{e} \lim_{n\to\infty} \left( \frac{\frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right)}{\frac{1}{n+1} - \frac{1}{n}} \right)$$

$$n \to \frac{1}{y} \ as \ n \to \infty, \ so, \ y \to 0$$

Now,

$$\Rightarrow \frac{2}{e} \lim_{y\to 0} \left( \frac{\frac{y}{y+1} - \ln(y+1)}{\frac{y}{1+y} - y} \right)$$

Applying L'Hospital's Rule, we get,

$$\Rightarrow \frac{2}{e} \lim_{y\to 0} \left(\frac{1}{2}\right)$$

(OR)

$$L = \frac{1}{e}$$

Shivam Sharma

**Fourth solution.** Let $h_n := \sum\limits_{k==1}^{n} \frac{1}{k}$. Since $\frac{1}{2(n+1)} < h_n - \ln n - \gamma < \frac{1}{2n}$ [1], then by Squeeze Principle

$$\lim_{n\to\infty} n(\gamma_n - \gamma) = \lim_{n\to\infty} n(h_n - \ln n - \gamma) = \frac{1}{2}.$$

Also, noting that $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ and $(2n-1)!! = \frac{(2n-1)!}{2^n n!}$ we obtain

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\sqrt[n]{(2n-1)!}}{2\sqrt[n]{n!}} =$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!}}{n^2} \cdot \frac{n}{\sqrt[n]{n!}} = \frac{e}{2} \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!}}{n^2} =$$

$$= \frac{e}{2} \lim_{n\to\infty} \left( \frac{\sqrt[2n]{(2n-1)!}}{n} \right)^2 = \frac{e}{2} \left( \lim_{n\to\infty} \frac{\sqrt[2n]{(2n)!}}{2n} \cdot \frac{2}{\sqrt[2n]{2n}} \right)^2 =$$

$$= \frac{e}{2} \left( \lim_{n\to\infty} \frac{\sqrt[2n]{(2n)!}}{2n} \cdot \frac{2}{\lim\limits_{n\to\infty} \sqrt[2n]{2n}} \right)^2 = \frac{e}{2} \cdot \left(\frac{1}{e} \cdot 2\right)^2 = \frac{2}{e}.$$

Hence,

$$\lim_{n\to\infty}(\gamma_n-\gamma)\sqrt[n]{(2n-1)!!}=\lim_{n\to\infty}n(\gamma_n-\gamma)\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=$$

$$=\lim_{n\to\infty}n(\gamma_n-\gamma)\cdot\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{1}{2}\cdot\frac{2}{e}=\frac{1}{e}.$$

1. R. M. Young, Euler's constant, Math. Gaz. 75 (1991), 187–190.

Arkady Alt

**Fifth solution.** We show that

$$\lim_{n\to\infty}n(\gamma_n-\gamma)=\frac{1}{2}$$

Indeed by using Cesaro–Stolz theorem we get

$$\lim_{n\to\infty}n(\gamma_n-\gamma)=\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}}=$$

$$=\lim_{n\to\infty}-n(n+1)\left(\frac{1}{n+1}-\ln(n+1)+\ln n\right)=$$

$$=\lim_{n\to\infty}-n(n+1)\left(\ln\left(1-\frac{1}{n+1}\right)+\frac{1}{n+1}\right)=$$

$$=\lim_{n\to\infty}-n(n+1)\left(\frac{-1}{2(n+1)^2}+o(\frac{1}{n^2})\right)=\frac{1}{2}$$

so

$$\lim_{n\to\infty}(\gamma_n-\gamma)\sqrt[n]{(2n-1)!!}=\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{1}{2}\lim_{n\to\infty}\left(\frac{(2n-1)!}{n^n2^{n-1}(n-1)!}\right)^{\frac{1}{n}}$$

Cesaro–Stolz again yields

$$\lim_{n\to\infty}\left(\frac{(2n-1)!}{n^n2^{n-1}(n-1)!}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{(2n+1)!}{(n+1)^{n+1}2^n n!}\frac{n^n2^{n-1}(n-1)!}{(2n-1)!}=$$

$$=\lim_{n\to\infty}\frac{(2n+1)!}{(2n-1)!}\frac{2^{n-1}}{2^n}\frac{(n-1)!}{n!}\frac{n^n}{(n+1)^{n+1}}=\lim_{n\to\infty}(2n+1)(2n)\frac{1}{2}\frac{1}{n}\frac{n^n}{(n+1)^{n+1}}$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2}{e}$$

whence the limit